THE CALGARY MATHEMATICAL ASSOCIATION

28^{TH} JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

April 28, 2004

NAME: SOLUTIONS GENDER: ☐ M ☐ F

PLEASE PRINT (First name Last name)

SCHOOL: ____________________ GRADE: ________ (7,8,9)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct.

  PART A has a total possible score of 45 points.

  PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.
THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

MARKERS' USE ONLY							
PART A ________ $\times 5$	B1	B2	B3	B4	B5	B6	TOTAL (max: 99)

PART A: SHORT ANSWER QUESTIONS

A1 Notice that $\frac{1}{2}+\frac{2}{4}=1$. Find the number N so that $\frac{2}{3}+\frac{3}{N}=1$.

9

A2 You have two triangles, which altogether have six angles. Five of these angles are 50°, 60°, 70°, 80°, and 90°. How large (in degrees) is the sixth angle?

10

A3 Pianos on the planet Zoltan have more keys than those on Earth, but otherwise are quite similar. The lowest white key on a Zoltan piano is A and the highest is C. In between, the white keys follow the repeating pattern $ABCDEFG$ and then starting over with A, eventually ending on C, just like on Earth. Which of the following numbers might be the number of white keys on a Zoltan piano?

100, 101, 102, 103, 104, 105, 106

101

A4 When Phillipa is born, her parents buy candles shaped like the ten digits 0 to 9. They buy **two** of each kind of candle. On each of Phillipa's birthdays they light the appropriate candles on her birthday cake. So, for example, on her first birthday they use just one "1" candle, while on her 10th birthday they use two candles, a "1" and a "0". Each candle can be used only 6 times altogether. Eventually there comes a birthday when both copies of one of the required candles are already used up. How old (in years) does Phillipa become on that birthday?

21

A5 Suppose you increase one side of a rectangle by 100% and the other side by 50%. By what percentage is the area of the rectangle increased?

200

A6. Beth buys \$9 worth of oranges that sell for \$0.75 each on Monday. On Thursday she finds that the oranges are on sale at \$0.25 each and buys another \$9 worth. What is the average cost per orange of the total number she bought?

37.5 cents

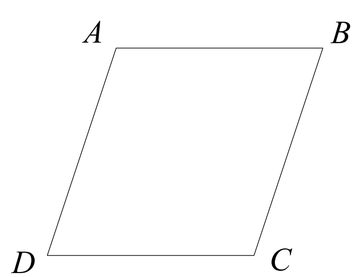
A7 Sam thinks of a number, and whispers it to Sabrina. Sabrina either adds two to the number or doubles it, and whispers the result to Susan. Susan takes that number and either subtracts 3 or divides the number by 3. The final result she announces is 10. What is the **largest** number Sam may have given Sabrina?

28

A8 Mr. Smith pours a full cup of coffee and drinks $\frac{1}{2}$ cup of it, deciding it is too strong and needs some milk. So he fills the cup with milk, stirs it, and tastes again, drinking another $\frac{1}{4}$ cup. Once again he fills the cup with milk, stirs it, and finds that this is just as he likes it. What ratio $\dfrac{\text{amount of coffee}}{\text{amount of milk}}$ does Mr. Smith like?

3/5

A9 In the figure $ABCD$ all four sides have length 10 and the area is 60. What is the length of the shorter diagonal, AC?



$\sqrt{40}$
$= 2\sqrt{10}$
≈ 6.32

PART B: LONG ANSWER QUESTIONS

B1 Zoar goes into a bookstore to buy a certain textbook. He has a discount card that will get a 10% discount on the book if he buys just one. However, if he buys a second, cheaper, paperback whose regular price (with GST) is \$6, the 10% discount will apply to the paperback and his discount card will give him a 20% discount on the textbook. He would pay the same total amount buying both books as buying the textbook alone. What is the regular price (with GST) of the textbook?

SOLUTION:

If Zoar buys the paperback, he pays \$6 minus the 10% discount for it, which comes out to be \$5.40. Since his total cost is the same as if he only bought the textbook, the extra 10% discount he gets on the textbook by buying the paperback too must save him exactly \$5.40. Therefore the regular price of the textbook must be **\$54**

Of course this problem could also be solved by algebra or by guess and check.

B2 Ioana writes down a 4 digit number $abcd$, any of whose digits, including a, may be a zero. She then calculates its "layer sum" by adding the 4 digit number $abcd$, the three digit number bcd, the two digit number cd, and the single digit d. (For example, the "layer sum" of the 4 digit number 0102 is $0102 + 102 + 02 + 2 = 208$.) What 4 digit number could she write down to produce a layer sum of 2004? Find all possible answers.

SOLUTION:

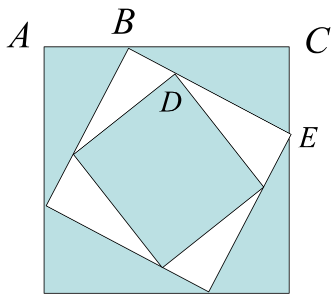
To get a layer sum of 2004, four of the digit d's added together must result in the digit 4, which means that d must be 1 or 6. Also, the digit a must be 2, 1 or 0.

If $a = 2$, then it is easy to get the answer **2001** , whose layer sum is $2001 + 001 + 01 + 1 = 2004$.

If $a = 1$, then $1bcd + bcd + cd + d = 2004$, so $bcd + bcd + cd + d = 1004$, so the digit b must be 4 or 5. If $b = 5$ then c must be 0 and d must be 1. Thus another answer is **1501** , whose layer sum is $1501 + 501 + 01 + 1 = 2004$. But if $b = 4$ then we get $4cd + 4cd + cd + d = 1004$, which means $cd + cd + cd + d = 204$. If $d = 1$ we would need three c's to add up to 20, which is impossible. So $d = 6$, which means that the four d's add up to 24, so the three c's must add up to 18, so $c = 6$. Thus a third answer is **1466** , whose layer sum is $1466 + 466 + 66 + 6 = 2004$.

If $a = 0$, then $0bcd + bcd + cd + d = 2004$, so b must be 9, and then $9cd + 9cd + cd + d = 2004$ means that $cd + cd + cd + d = 204$. Just like in the previous case, the only way this can happen is if $d = 6$ and $c = 6$, so we get a fourth answer of **0966** , whose layer sum is $0966 + 966 + 66 + 6 = 2004$.

B3 You have three inscribed squares, with the corners of each inner square at the $\frac{1}{4}$ point along the sides of its outer square. (So, for example, $AB = \frac{1}{4}AC$, and $BD = \frac{1}{4}BE$.) The area of the largest square is 64 m^2. What is the area of the smallest square?



SOLUTION:

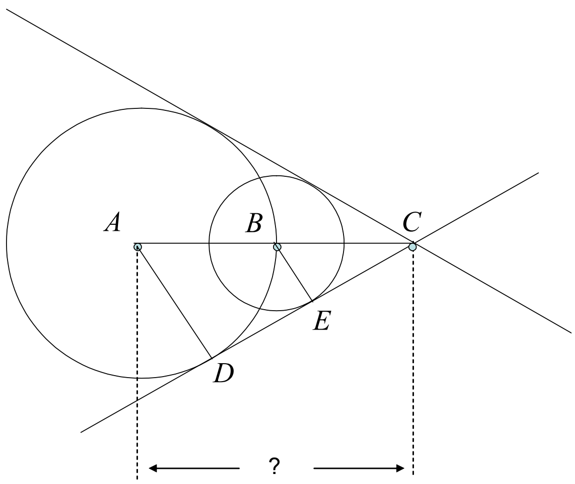
Since the area of the largest square is 64, $AC = \sqrt{64} = 8$. Since $AB = \frac{1}{4}AC$, $AB = 2$, so $BC = 6$. Similarly $CE = 2$. By the Pythagorean Theorem,

$$BE = \sqrt{BC^2 + CE^2} = \sqrt{6^2 + 2^2} = \sqrt{40}.$$

This says that the area of the middle square is 40, which is $40/64 = 5/8$ the area of the outside square. Since the inside (smallest) square must have the same relation to the middle square, the area of the smallest square must be $5/8$ the area of the middle square, that is, the area of the smallest square is $(5/8) \times 40 = \mathbf{25}$.

B4 The centre of a circle of radius 1 cm lies on the circumference of a circle of radius 3 cm. How far (in cm) from the centre of the big circle do the common tangents of the two circles meet?

SOLUTION:

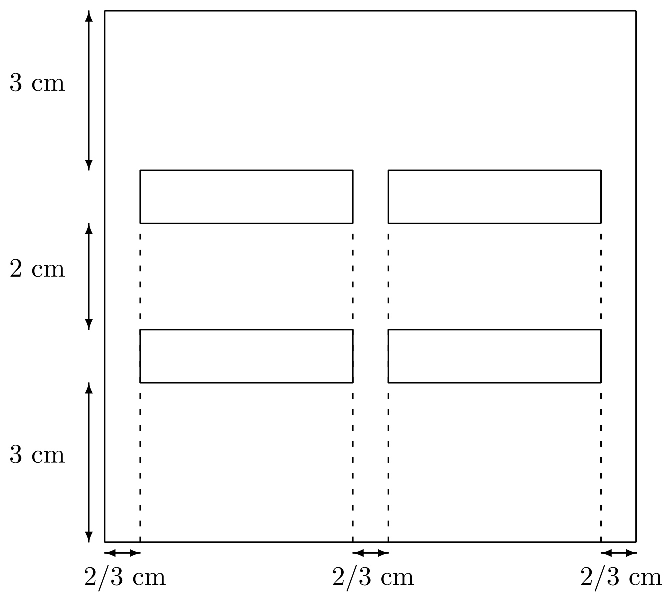


Label the diagram as shown. Then $AD = 3$ and $BE = 1$. So the two triangles ACD and BCE are similar and the sides of ACD must be three times as big as the corresponding sides of BCE. This means that AC is three times as long as BC, so AB must be twice as long as BC. But AB is a radius of the big circle, so $AB = 3$. Therefore $BC = 3/2$. Thus $AC = AB + BC = 3 + 3/2 = \mathbf{9/2}$.

B5 You have a 10 cm by 10 cm square whose sides are horizontal and vertical, and a large supply of rectangular strips of cardboard 1 cm by 4 cm. You want to place some of the cardboard strips horizontally inside the square so that it is impossible to place one of the 1 cm by 4 cm strips **vertically** inside the square without overlapping at least one of the horizontal strips. Show how to place the horizontal strips to do this. The fewer horizontal strips you need to use, the better your answer is. Make sure to describe exactly where the strips go.

SOLUTION:

Here is a way to place **four** horizontal strips in the square so that no vertical strip can be placed without overlapping:



There are other good ways to place the four strips, but three horizontal strips would not be enough.

B6 Each year, Henry's parents give him some money on his birthday, calculated as follows: they give him a number of pennies equal to his birth year, a number of dimes equal to the day of the month he was born, a number of quarters equal to the month he was born in (1 quarter for January, 2 quarters for February, and so on), and a number of loonies equal to his age. (So, for example, if Henry had been born on November 14, 1972, on his birthday in 2003 he would have received 1972 pennies, plus 14 dimes, plus 11 quarters, plus 31 loonies for a total of $\$19.72 + \$1.40 + \$2.75 + \$31 = \$54.87$.)

Actually, on his birthday in 2003 Henry received \$32.96. Find all possibilities for Henry's date of birth (day, month, and year).

SOLUTION:

First, the only way to get a number of pennies which is not a multiple of 5 is through the birth year. Since Henry received \$32.96, his birth year must end in a 1 or a 6. So it must be 1991 or 1996, as other years ending in 1 or 6 result in either too much or too little money.

For 1991, Henry would have turned 12 in 2003, so he would get \$19.91 for his birth year and \$12 for his age, so he would have to get $32.96 - 19.91 - 12 = \$1.05$ for his day and month of birth. Months give quarters and days give dimes, so he would have to receive either one or three quarters to account for the 5 cents. If he gets only 1 quarter then he would need 8 dimes, which means his date of birth would be **January 8, 1991** . If he gets 3 quarters, then he would need 30 cents more or 3 dimes, which means his date of birth would be **March 3, 1991** .

For 1996, Henry would have turned 7 in 2003, so he would get \$19.96 for his birth year and \$7 for his age, so he would have to get $32.96 - 19.96 - 7 = \$6$ for his day and month of birth. The most number of quarters he could get is 12 (if he were born in December), and these only amount to \$3, so he would need 30 dimes to make up the \$6. This is possible, and it means his date of birth would be **December 30, 1996** .